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# CONCEPTUAL DESIGN OF SCHOENFLIES MOTION GENERATORS BASED ON THE WRENCH GRAPH

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## ABSTRACT

*The subject of this paper is about the conceptual design of parallel Schoenflies motion generators based on the wrench graph.*

*By using screw theory and Grassmann geometry, some conditions on both the constraint and the actuation wrench systems are generated for the assembly of limbs of parallel Schoenflies motion generators, i.e., 3T1R parallel manipulators. Those conditions are somehow related to the kinematic singularities of the manipulators. Indeed, the parallel manipulator should not be in a constraint singularity in the starting configuration for a valid architecture, otherwise it cannot perform the required motion pattern. After satisfying the latter condition, the parallel manipulator should not be in an actuation singularity in a general configuration, otherwise the obtained parallel manipulator is permanently singular.*

*Based on the assembly conditions, six types of wrench graphs are identified and correspond to six typical classes of 3T1R parallel manipulators. The geometric properties of these six classes are highlighted. A simplified expression of the superbracket decomposition is obtained for each class, which allows the determination and the comparison of the singularities of 3T1R parallel manipulators at their conceptual design stage. The methodology also provides new architectures of parallel Schoenflies motion generators based on the classification of wrench graphs and on their singularity conditions.*

## INTRODUCTION

The primary concern of the conceptual design is the generation of physical solutions to meet certain design specification [1]. The concept generated at this phase affects the basic shape generation and material selection of the product concerned. In the subsequent phase of detailed design, it becomes exceedingly difficult, or even impossible to make a correction the shortcomings of a poor design concept formulated at the conceptual design stage [2]. Therefore, the conceptual design of parallel manipulators (PMs) is crucial task which aims at defining the architectures of the associated kinematic chains. In this paper, the focus of conceptual design process is the Schoenflies Motion Generators (SMGs).

The parallel manipulators are called Schoenflies Motion Generators if they can perform four degree of freedom (*dof*) displacements of a rigid body. These motions involve three independent translations and one rotation about a fixed axis [3]. This set

of displacements was first studied by the German mathematician-mineralogist Arthur Moritz Schoenflies (1853-1928).

Over the past few decades, the creation of various designs of 3T1R parallel manipulators were broaden, especially after huge success of the Quattro [4]. Gogu discovered an isotropic architecture (its jacobian matrix is diagonal and constant), named the Isoglide4 [5, 6], which is composed of four legs with prismatic actuators. Another topology within the same family was introduced by Gosselin [7], named the Quadrupton.

The symmetrical design was proposed by Angeles [3], namely the McGill SMG, with two identical legs which in turn decreases the number of joints. There exist two architecture varieties of H4 family, either with revolute or prismatic actuators [8–10]. The H4 robot designed by Pierrot *et al.* [8, 9], is a fully-parallel mechanism with no passive kinematic chain between the base and the nacelle. This idea brought out the mechanism with four legs. Each revolute joint in the leg is actuated. Whereas the H4 robot with prismatic actuators mounted to the base, was presented by Wu *et al.* [10]. Another mechanism constructed by three identical legs was proposed by Briot and Bonev [11, 12], called Pantopteron-4, where each leg comprises a pantograph linkage. Since it only employs three legs, the Pantopteron-4 gains great advantage in terms of workspace volume and acceleration capacities.

The type synthesis approach based upon screw theory is widely used for generating many parallel manipulators as shown by Kong and Gosselin [13]. This approach allows us to produce numerous kinematic chains, by discovering the wrench system  $\mathcal{W}$  that is reciprocal to the twist system  $\mathcal{T}$  of the moving platform.

Based upon the reciprocity condition, Joshi and Tsai developed a procedure to express Jacobian matrix  $\mathbf{J}$  of limited *dof* parallel manipulators, comprises both constraint and actuation wrenches [14]. In this paper, this matrix is named the extended Jacobian matrix  $\mathbf{J}_E$ . The rows of  $\mathbf{J}_E$  are composed of  $n$  linearly independent actuation wrenches plus  $(6-n)$  linearly independent constraint wrenches.

These wrenches correspond to six Plücker lines, composing  $\mathbf{J}$ . The determinant of this  $\mathbf{J}$  is equal to the superjoin of 6 Plücker lines, named superbracket of Grassmann-Cayley Algebra. It allows a translation of synthetic geometric conditions into invariant (coordinate-free) algebraic expression [15].

The superbracket decomposition was employed by Ben-Horin and Shoham [16–18] to analyse the singularity of 6 *dof*

parallel manipulators, for instance Gough-Stewart Platforms. However, this procedure does not consider the plane at infinity, which obviously cannot represent a line at infinity. Indeed, a line at infinity correlates to a pure moment emerged from limited *dof* parallel manipulators.

Points at infinity were used initially in a superbracket to examine the singularity of 3-UPU by Kanaan et al [19, 20]. This improvement enhanced the application of Grassmann-Cayley Algebra for limited *dof* parallel manipulators. The method was later expanded to represent the wrenches in a projective space, named *wrench graph* by Amine et al [21–23]. The wrench graph depicts all geometrics properties between the constraint and actuation wrenches of manipulators and highlight points at infinity in a superbracket.

The concept of wrench graph consequently completes the type synthesis process (based on screw theory) at the conceptual design stage. This idea will be applied in this paper to illustrate all the constraint and actuation wrenches. It allows to construct new architectures of 3T1R parallel manipulators based on the classification of wrench graph and the singularity conditions.

## TYPE SYNTHESIS OF 3T1R PMs

The moving platform of a 3T1R PM provides three independent translational *dof* and one rotational *dof* about an axis of fixed direction. Without loss of generality, we consider that the rotational *dof* is about an axis directed along  $\mathbf{z}$ .

### Constraint Wrench System

In a general configuration, the constraint wrench system,  $\mathcal{W}_c$ , of a 3T1R PM must be reciprocal to the 3T1R motion. Thus, it is a 2- $\infty$ -system containing infinite-pitch wrenches (pure moments) whose directions are orthogonal to  $\mathbf{z}$ . Accordingly,  $\mathcal{W}_c$  can be written as:

$$\mathcal{W}_c = \text{span}(\mathcal{M}_{c1}, \mathcal{M}_{c2}) \quad (1)$$

where  $\mathcal{M}_{c1} = (\mathbf{0}, \mathbf{z} \times \mathbf{m}_1)$ ,  $\mathcal{M}_{c2} = (\mathbf{0}, \mathbf{z} \times \mathbf{m}_2)$  and  $\mathbf{z}$ ,  $\mathbf{m}_1$  and  $\mathbf{m}_2$  are three mutually independent unit vectors.

The constraint moment  $\mathcal{M}_{c1}$  corresponds to a line at infinity passing through points  $\underline{j} = (\mathbf{z}, 0)$  and  $\underline{i} = (\mathbf{m}_1, 0)$ . Thus,  $\mathcal{M}_{c1} = \underline{j}\underline{i}$ . Likewise,  $\mathcal{M}_{c2} = \underline{j}\underline{k}$  where  $\underline{k} = (\mathbf{m}_2, 0)$ . As a result, the constraint wrench system  $\mathcal{W}_c = \text{span}(\mathcal{M}_{c1}, \mathcal{M}_{c2})$  corresponds (in a general configuration) to a flat pencil containing all lines at infinity through point  $\underline{j} = (\mathbf{z}, 0)$ ,  $\mathbf{z}$  being parallel to the axis of the allowed rotation of the PM's moving platform.

**Constraint Singularities** The constraint singularities of a 3T1R PM correspond to configurations in which the constraint wrench system fails to be a 2-system corresponding to a flat pencil of lines at infinity through point  $\underline{j}$ . In such configurations, the PM can switch to another motion mode whose motion pattern is no longer a 3T1R motion.

However, if the constraint wrench system of a 3T1R PM corresponds to the 2-system in any configuration, then the PM will be free of constraint singularities. Thus, it is possible to

avoid such configurations with an appropriate choice of the PM's architecture, as shown thereafter in this paper.

### Actuation Wrench System

In a general configuration, by locking all the actuated joints of a PM its moving platform must be fully constrained, i.e., its actuation wrench system must span in addition to the constraint wrench system a 6-system. Thus, the actuation wrench system  $\mathcal{W}_a$  of a 3T1R PM must be (in a general configuration) a 4-system and the linear combination of  $\mathcal{W}_c$  and  $\mathcal{W}_a$  must lead to a 6-system, otherwise the robot is permanently singular.

**Actuation Singularities** The actuation singularities occur when the actuation wrench system of the PM does not satisfy any longer the previous condition while the PM is not in a constraint singularity. Thus, a 3T1R PM exhibits an actuation singularity whenever: 1.  $\mathcal{W}_a$  fails to be a 4-system; and 2. the linear combination of  $\mathcal{W}_a$  and  $\mathcal{W}_c$  fails to be a 6-system while  $\mathcal{W}_c$  is a 2-system.

### Comparison of Limbs for 3T1R PMs

A list of 4- and 5- *dof* limbs for 3T1R PMs with R- and/or P-joints was obtained in [13]. Without loss of generality, we consider the rotational *dof* of the 3T1R PMs under study is about an axis directed along  $\mathbf{z}$ . Accordingly, the following conditions should necessarily be satisfied:

- In a given limb, the axes of the R-joints must lie in parallel planes;
- If the axes of all the R-joints in a limb are parallel, then they will be necessarily directed along  $\mathbf{z}$ ;

In order to highlight the axis direction of a revolute joint, let:

1.  $\bar{\mathbf{R}}$  denote a revolute joint of axis parallel to  $\mathbf{z}$ ;
2.  $\bar{\mathbf{R}}$  can be:
  - (a) a revolute joint,  $\bar{\mathbf{R}}$ , of axis parallel to the horizontal plane ( $\mathbf{xOy}$ );
  - (b) a revolute joint  $\bar{\mathbf{R}}$  whose axis is neither parallel to  $\mathbf{z}$  nor to the horizontal plane ( $\mathbf{xOy}$ ).
3. In the  $i$ th limb, let  $\mathbf{m}_i$  denote the unit vector along the direction of the revolute joint axes that are not parallel to  $\mathbf{z}$ ;

Therefore, all  $\bar{\mathbf{R}}$ -joints (all  $\bar{\mathbf{R}}$ -joints, respectively) in a given limb have parallel axes. Now let us reformulate the list of limbs for 3T1R PMs obtained in [Kong] under the above assumptions. Accordingly, Tab. 1 is obtained. Here, we examine this list by a comparative analysis of the limbs in terms of both constraint and actuation wrench systems.

### Limbs Constraint Wrench System

The constraint wrench system of a given limb only depends on the joint types and the joint axes. However, the arrangement of the joints within the limb may affect the constraint singularities of the obtained 3T1R PM.

**TABLE 1.** LIMBS FOR 3T1R PMs.

dof	Class	Type
4	3R-1P	Permutation of $\bar{P}\bar{R}\bar{R}\bar{R}$
	2R-2P	Permutation of $\bar{P}\bar{P}\bar{R}\bar{R}$
	1R-3P	Permutation of $\bar{P}\bar{P}\bar{P}\bar{R}$
5	5R	Permutation of $\bar{R}\bar{R}\bar{R}\bar{R}\bar{R}$
		Permutation of $\bar{R}\bar{R}\bar{R}\bar{R}\bar{R}$
	4R-1P	Permutation of $\bar{P}\bar{R}\bar{R}\bar{R}\bar{R}$
		Permutation of $\bar{P}\bar{R}\bar{R}\bar{R}\bar{R}$
		Permutation of $\bar{P}\bar{R}\bar{R}\bar{R}\bar{R}$
	3R-2P	Permutation of $\bar{P}\bar{P}\bar{R}\bar{R}\bar{R}$
		Permutation of $\bar{P}\bar{P}\bar{R}\bar{R}\bar{R}$
	2R-3P	Permutation of $\bar{P}\bar{P}\bar{P}\bar{R}\bar{R}$

**Four dof Limbs** Any 4-dof limb of Tab. 1 applies a 2- $\mathcal{S}_\infty$ -system of constraint wrenches reciprocal to the  $\bar{R}$ -joints of the limb that are directed along  $\mathbf{z}$ . Thus, such a limb applies, in any configuration, an infinite number of constraint moments that correspond to lines at infinity passing through point  $\underline{j} = (\mathbf{z}, 0)$ . It can be concluded that a 3T1R PM having at least one 4-dof limb of Tab. 1 is over-constrained and free of constraint singularity.

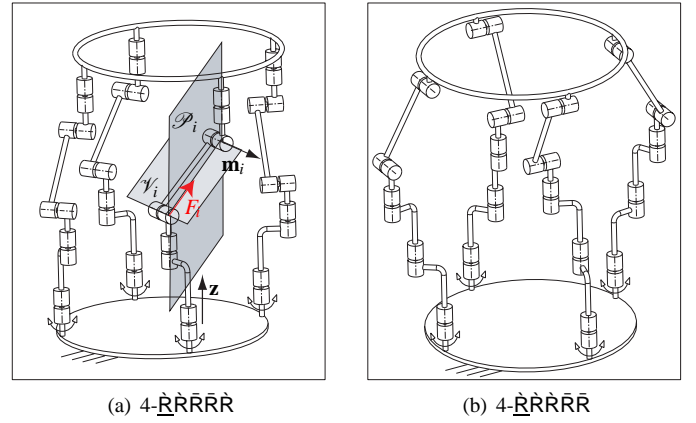
It should be noted that for the 4-dof limbs given in Tab. 1 the directions of the  $\bar{P}$ -joints are defined in such a way that the limb generates a 3T1R motion. For example, let us consider the  $\bar{P}\bar{R}\bar{R}\bar{R}$  limb type. The three  $\bar{R}$ -joints generate, in a general configuration, one rotation about an axis directed along  $\mathbf{z}$  and two translations along vectors orthogonal to  $\mathbf{z}$ . Thus, the  $\bar{P}$ -joint is necessarily along  $\mathbf{z}$ , otherwise the limb cannot provide the required 3-dof translational motion.

**Five dof Limbs** A 5-dof limb used in the topology of a 3T1R PMs must apply one constraint moment that corresponds to a line at infinity passing through point  $\underline{j} = (\mathbf{z}, 0)$ . Accordingly, the 3T1R PM must be assembled such that any 5-dof limb of the PM provides a 3T2R motion in the starting configuration. The 5-dof limbs given in Tab. 1 can be classified into two types:

**Type 1** There exist within the limb two revolute joints of parallel axes that are neither successive nor separated by a  $\bar{P}$ -joint. Such a limb is not generally a 3T2R limb. The 3T1R PM containing limbs of this type is obtained by imposing some geometric constraints on the assembly.

**Type 2** Two revolute joints of parallel axes within the limb are either successive or separated by a  $\bar{P}$ -joint. Such a limb is generally a 3T2R limb.

In order to highlight the difference between the two types of limbs let us consider two 3T1R PMs with identical limb structures, namely, a 4- $\bar{R}\bar{R}\bar{R}\bar{R}\bar{R}$  PM and a 4- $\bar{R}\bar{R}\bar{R}\bar{R}\bar{R}$  PM shown in Fig. 1.



**FIGURE 1.** TWO 3T1R PMs.

Clearly, for a  $\bar{R}\bar{R}\bar{R}\bar{R}\bar{R}$  limb, the last revolute joint axis is not necessarily parallel to the first two. Thus, the 4- $\bar{R}\bar{R}\bar{R}\bar{R}\bar{R}$  3T1R PM is obtained by assembling the PM such that:

- the condition of parallelism between the last revolute joint axis and the first two is satisfied for all limbs;
- vectors  $\mathbf{m}_i$ ,  $i = 1, \dots, 4$ , are not all parallel.

In that case, the limbs apply altogether four constraint wrenches that correspond to four lines at infinity passing through point  $\underline{j}$  and span a 2-system of constraint wrenches. However, if the four vectors  $\mathbf{m}_i$  become parallel, the PM will exhibit a constraint singularity and can switch to another motion mode in which the motion of the moving platform will no longer be a 3T1R motion. In such a configuration, the last revolute joint axis in each limb of the 4- $\bar{R}\bar{R}\bar{R}\bar{R}\bar{R}$  PM will no longer be parallel to  $\mathbf{z}$  and the moving platform will no longer be parallel to the horizontal plane ( $\mathbf{xOy}$ ).

Now let us consider the 4- $\bar{R}\bar{R}\bar{R}\bar{R}\bar{R}$  PM. Clearly, its limbs do not need geometric assembly conditions in order to provide a 3T2R motion for the limb and a 3T1R motion for the moving platform. This PM is free of constraint singularities and the moving platform is always parallel to plane ( $\mathbf{xOy}$ ). Moreover, it can be concluded that if a 3T1R PM contains at least one limb of type 2, its moving platform will always be parallel to plane ( $\mathbf{xOy}$ ) and thus, the PM will be free of constraint singularities. However, if all the limbs of the 3T1R PM are of type 1, then the PM may have some constraint singularities.

### Limbs Actuation Wrench System

For a given type of limbs given in Tab. 1, the actuation wrenches do not depend on the arrangement of the joints within the limb but on the actuated joints. The constraint wrench system  $\mathcal{W}^c$  of a limb is reciprocal to all the twists associated with the limb's kinematic joints. If the actuated joint of the limb is locked, then the wrench system reciprocal to the other joints of the limb will be given by  $\mathcal{U}^c$  such that:  $\dim(\mathcal{U}^c) = \dim(\mathcal{W}^c) + 1$  and  $\mathcal{U}^c$  contains  $\mathcal{W}^c$  plus a set of some additional wrenches. Then, the corresponding actuation wrench can be selected as one of these additional wrenches.

Table 2 characterizes the locus of the actuation wrench for each type of limbs obtained in Tab. 1 and for each type of actu-

ated joints. Without loss of generality, we consider that one joint per limb is actuated.

### Validity Rule of the Actuation Wrench for 3T1R PM

In what follows, let us assume that the limbs apply a 2-system of constraint wrenches spanned by  $\mathcal{M}_{c1}$  and  $\mathcal{M}_{c2}$  and satisfying Eqn. (1). In this section, we develop some rules to examine the validity of the actuation wrench system of a 3T1R PM. For instance, we consider that an actuation wrench applied by a given limb of a PM is a zero- or an infinite-pitch wrench namely, a pure force or a pure moment. Let  $\mathcal{F}_1, \dots, \mathcal{F}_4$  denote actuation forces and  $\mathcal{M}_a, \mathcal{M}_{a1}$  and  $\mathcal{M}_{a2}$  denote actuation moments. In a general configuration of a 3T1R PM, one can state the following rules.

**R1**<sup>1</sup> A basis of the actuation wrench system  $\mathcal{W}^a$  can contain at most one actuation moment.

**Proof** Let us consider that a basis of  $\mathcal{W}^a$  contains two actuation moments  $\mathcal{M}_{a1}$  and  $\mathcal{M}_{a2}$ . In that case,  $\mathcal{W}^a + \mathcal{W}^c = \text{span}(\mathcal{F}_1, \mathcal{F}_2, \mathcal{M}_{a1}, \mathcal{M}_{a2}, \mathcal{M}_{c1}, \mathcal{M}_{c2})$ . Since  $\dim(\text{span}(\mathcal{M}_{a1}, \mathcal{M}_{a2}, \mathcal{M}_{c1}, \mathcal{M}_{c2})) \leq 3$ , the dimension of  $\mathcal{W}^a + \mathcal{W}^c$  will be lower than or equal to 5 in any robot configuration and thus the proposed PM will be permanently singular.

**R2** If a basis of  $\mathcal{W}^a$  contains one actuation moment  $\mathcal{M}_a$ , then the line at infinity corresponding to  $\mathcal{M}_a$  should not pass through point  $\underline{j} = (\mathbf{z}, 0)$ .

**Proof** Let us assume that a basis of  $\mathcal{W}^a$  contains one actuation moment  $\mathcal{M}_a$  that corresponds to a line at infinity passing through point  $\underline{j}$ , in a general configuration. In that case, the actuation moment  $\mathcal{M}_a$  belongs to the constraint wrench system in a general configuration. Clearly in that case:  $\dim(\mathcal{W}^a + \mathcal{W}^c) \leq 5$ .

**R3** If the basis of  $\mathcal{W}^a$  contains one actuation moment  $\mathcal{M}_a$  and three actuation forces  $\mathcal{F}_1, \dots, \mathcal{F}_3$ , then none of the three actuation forces can be parallel to another.

**Proof** Indeed, if  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are parallel, then  $\text{span}(\mathcal{F}_1, \mathcal{F}_2)$  is equivalent to  $\text{span}(\mathcal{F}_1, \mathcal{M}_{12})$ ,  $\mathcal{M}_{12}$  being the line at infinity of the finite plane containing  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . Accordingly, one can find a basis of  $\mathcal{W}^a$  containing two actuation moments, and thus, rule **R1** is not satisfied in that case.

**R4** If a basis of  $\mathcal{W}^a$  contains one actuation moment  $\mathcal{M}_a$  and three actuation forces  $\mathcal{F}_1, \dots, \mathcal{F}_3$ , then the three actuation forces cannot be coplanar.

**Proof** Indeed, if  $\mathcal{F}_1, \mathcal{F}_2$  and  $\mathcal{F}_3$  are coplanar, then  $\text{span}(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$  is equivalent to  $\text{span}(\mathcal{F}_1, \mathcal{F}_2, \mathcal{M}_{a1})$ ,  $\mathcal{M}_{a1}$  being the line at infinity of the finite plane containing  $\mathcal{F}_i$  ( $i = 1, \dots, 3$ ). Accordingly, rule **R1** is not satisfied in that case.

**R5** If a basis of  $\mathcal{W}^a$  is composed of four actuation forces, then it can contain at most one pair of parallel actuation forces, provided that their plane does not contain the unit vector  $\mathbf{z}$ .

**Proof** Clearly, if a basis of  $\mathcal{W}^a$  contains more than one pair of parallel actuation forces, then one can compute another basis of  $\mathcal{W}^a$  with more than one actuation moment, and thus, rule **R1** will not be satisfied.

**R6** If a basis of  $\mathcal{W}^a$  is composed of four actuation forces where two forces  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are parallel, then the plane containing these forces must not contain the unit vector  $\mathbf{z}$ .

**Proof** If the plane containing the two parallel actuation forces contains the unit vector  $\mathbf{z}$ , it follows that the two actuation forces  $\mathcal{F}_1$  and  $\mathcal{F}_2$  and the two constraint moments  $\mathcal{M}_{c1}$  and  $\mathcal{M}_{c2}$  will belong altogether to the union of two flat pencils of lines. Accordingly,  $\dim(\text{span}(\mathcal{F}_1, \mathcal{F}_2, \mathcal{M}_{c1}, \mathcal{M}_{c2})) \leq 3$  and the dimension of  $\mathcal{W}^a + \mathcal{W}^c$  will be lower than or equal to 5 in any robot configuration.

**R7** If a basis of  $\mathcal{W}^a$  is composed of four actuation forces, then at most three of these forces can be coplanar.

**Proof** Indeed, if the four forces are coplanar, then the actuation wrench system fails to be a 4-system in a general configuration, since four coplanar forces can span at most a 3-system.

**R8** A basis of  $\mathcal{W}^a$  can contain at most one actuation force parallel to  $\mathbf{z}$ .

**Proof** Indeed, if two actuation forces,  $\mathcal{F}_1$  and  $\mathcal{F}_2$ , are directed along  $\mathbf{z}$ , then  $\text{span}(\mathcal{F}_1, \mathcal{F}_2)$  is equivalent to  $\text{span}(\mathcal{F}_1, \mathcal{M}_{12})$  where  $\mathcal{M}_{12}$  corresponds to a line at infinity that necessarily passes through point  $\underline{j} = (\mathbf{z}, 0)$ . Accordingly, in that case, rule **R2** will not be satisfied.

**R9** If a basis of  $\mathcal{W}^a$  is composed of four actuation forces, these forces cannot be orthogonal to a common direction.

**Proof** Indeed, if the four actuation forces are (in a general configuration) orthogonal to a common direction, then they will intersect a common line at infinity, which necessarily intersects all the wrenches of the constraint wrench system. Thus, in that case, the four actuation forces plus the 2- $\infty$ -system of constraint wrenches form a singular complex.

## TWO MAIN CLASSES OF SMGs

Based on the properties presented in the previous section, two main classes of 3T1R PMs can be identified. For PMs of the first class, one can find (in a general configuration) a basis of the actuation wrench system that contains one actuation moment. On the other hand, a basis of the actuation wrench system of PMs of the second class can be composed only of actuation forces. By following rules **R1**...**R9**, some sub-classes of each class can be obtained.

For instance, we will represent the wrench graphs of three sub-classes of each class. However, by following the same procedure and satisfying rules **R1**...**R9**, other sub-classes can be obtained. It is noteworthy that each sub-class is characterized by the description of the actuation wrench system, the constraint wrench system being the same for any 3T1R PM.

1. Class A: The actuation wrench system contains one line at infinity.

**A.1** One actuation moment plus three actuation forces. In that case, the three actuation forces cannot be coplanar and none of these forces can be parallel to another;

**A.2** Two parallel actuation forces plus two other actuation forces;

<sup>1</sup> **R1** stands for rule 1

**A.3** Three coplanar actuation forces plus a fourth actuation force that does not belong to the plane of the first three ones.

A wrench graph for each sub-class is given in Tab. 3.

2. Class B: The actuation wrench system does not contain any line at infinity.

**B.1** Four actuation forces that are mutually skew where none of the four forces is parallel to  $\mathbf{z}$ ;

**B.2** Four actuation forces that are mutually skew where one of the four forces is parallel to  $\mathbf{z}$ ;

**B.3** Two pairs of concurrent actuation forces.

A wrench graph for each sub-class is given in Tab. 3.

### Superbracket Decomposition for Class A

#### A.1- One actuation moment plus three actuation forces

Let us consider a PM of class A.1. A basis of the actuation wrench system  $\mathcal{W}^a$  of the PM can be composed of three actuation forces  $\hat{\mathcal{F}}_1$ ,  $\hat{\mathcal{F}}_2$  and  $\hat{\mathcal{F}}_3$  and one actuation moment  $\hat{\mathcal{M}}_a$ . Now let  $\hat{\mathcal{M}}_{c1}$  and  $\hat{\mathcal{M}}_{c2}$  be two constraint moments that form a basis of the constraint wrench system  $\mathcal{W}^c$  of the PM. Accordingly, the extended Jacobian matrix of this sub-class of SMGs takes the form:

$$\mathbf{J}_E^T = [\hat{\mathcal{F}}_1 \ \hat{\mathcal{F}}_2 \ \hat{\mathcal{F}}_3 \ \hat{\mathcal{M}}_a \ \hat{\mathcal{M}}_{c1} \ \hat{\mathcal{M}}_{c2}] \quad (2)$$

From rule **R1**, the actuation moment  $\hat{\mathcal{M}}_a$  must correspond to a line at infinity that does not pass through point  $\underline{j}$ , in a general configuration of the PM. Moreover, in a general configuration, the three actuation forces cannot be coplanar and none of these forces can be parallel to another. Let us consider the general case satisfying these conditions by considering that the three actuation forces are skew (in a general configuration). We know that three moments are always mutually concurrent. Thus, the six wrenches composing  $\mathbf{J}_E$  can be written as:  $\hat{\mathcal{F}}_1 = (\mathbf{f}_1, \mathbf{r}_{A_1} \times \mathbf{f}_1) = \underline{ab}$ ,  $\hat{\mathcal{F}}_2 = (\mathbf{f}_2, \mathbf{r}_{A_2} \times \mathbf{f}_2) = \underline{cd}$ ,  $\hat{\mathcal{F}}_3 = (\mathbf{f}_3, \mathbf{r}_{A_3} \times \mathbf{f}_3) = \underline{ef}$ ,  $\hat{\mathcal{M}}_a = (\mathbf{0}, \mathbf{m}_a) = \underline{ik}$ ,  $\hat{\mathcal{M}}_{c1} = \underline{ij} = (\mathbf{0}, \mathbf{m}_{c1})$ ,  $\hat{\mathcal{M}}_{c2} = \underline{kj} = (\mathbf{0}, \mathbf{m}_{c2})$ . The corresponding superbracket is then expressed and simplified as follows:

$$S = [\underline{ab} \ \underline{cd} \ \underline{ef} \ \underline{ik} \ \underline{ij} \ \underline{kj}] = [\underline{abdf}][\underline{cijk}][\underline{eikj}] \quad (3)$$

Such a PM has two singularity conditions:

- (a)  $(\mathbf{f}_1 \times \mathbf{f}_2) \cdot \mathbf{f}_3 = 0$
- (b)  $(\mathbf{m}_{c1} \times \mathbf{m}_{c2}) \cdot \mathbf{m}_a = 0$

#### A.2- Two parallel actuation forces plus two other skew actuation forces

The extended Jacobian matrix of this sub-class of SMGs takes the form:

$$\mathbf{J}_E^T = [\hat{\mathcal{F}}_1 \ \hat{\mathcal{F}}_2 \ \hat{\mathcal{F}}_3 \ \hat{\mathcal{F}}_4 \ \hat{\mathcal{M}}_{c1} \ \hat{\mathcal{M}}_{c2}] \quad (4)$$

where  $\hat{\mathcal{F}}_1$  is parallel to  $\hat{\mathcal{F}}_2$ . Let  $\hat{\mathcal{F}}_1 = (\mathbf{f}_1, \mathbf{r}_{A_1} \times \mathbf{f}_1) = \underline{ab}$ ,  $\hat{\mathcal{F}}_2 = (\mathbf{f}_1, \mathbf{r}_{A_2} \times \mathbf{f}_1) = \underline{cb}$ ,  $\hat{\mathcal{F}}_3 = (\mathbf{f}_3, \mathbf{r}_{A_3} \times \mathbf{f}_3) = \underline{ef}$  and  $\hat{\mathcal{F}}_4 = (\mathbf{f}_4, \mathbf{r}_{A_4} \times \mathbf{f}_4) = \underline{gh}$  be the four actuation forces. In turn, let  $\hat{\mathcal{M}}_{c1} = (\mathbf{0}, \mathbf{m}_{c1}) = \underline{ij}$  and  $\hat{\mathcal{M}}_{c2} = (\mathbf{0}, \mathbf{m}_{c2}) = \underline{kj}$  be the two constraint moments.

The corresponding superbracket is expressed and decomposed as:

$$\begin{aligned} S &= [\underline{ab} \ \underline{cb} \ \underline{ef} \ \underline{gh} \ \underline{ij} \ \underline{kj}] \\ &= [\underline{eikj}] \left( [\underline{abch}] [\underline{befj}] - [\underline{abcf}] [\underline{behj}] \right) \\ &= [\underline{eikj}] \left( [\underline{abch}] [\underline{befj}] \right) \\ &= [\underline{eikj}] \left( (\underline{abc}) \wedge (\underline{bej}) \wedge (\underline{hfi}) \right) \end{aligned} \quad (5)$$

Let  $\mathbf{u}_{ac}$  be the unit vector of the finite line joining the projective points a and c. This sub-class of SMGs has two main singularity conditions:

- (a)  $(\mathbf{m}_{c1} \times \mathbf{m}_{c2}) \cdot \mathbf{z} = 0$ . Clearly, this is the condition for constraint singularities.
- (b)  $((\mathbf{f}_1 \times \mathbf{u}_{ac}) \times (\mathbf{f}_1 \times \mathbf{z})) \cdot (\mathbf{f}_4 \times \mathbf{f}_3) = 0 \Rightarrow$ 
  - 1.  $\mathbf{f}_1 \parallel \mathbf{u}_{ac}$ ;
  - 2.  $\mathbf{f}_1 \parallel \mathbf{z}$ ;
  - 3.  $\mathbf{f}_3 \parallel \mathbf{f}_4$ .

#### A.3- Three coplanar actuation forces plus a fourth actuation force

The extended Jacobian matrix of this sub-class of SMGs takes the form:

$$\mathbf{J}_E^T = [\hat{\mathcal{F}}_1 \ \hat{\mathcal{F}}_2 \ \hat{\mathcal{F}}_3 \ \hat{\mathcal{F}}_4 \ \hat{\mathcal{M}}_{c1} \ \hat{\mathcal{M}}_{c2}] \quad (6)$$

where  $\hat{\mathcal{F}}_1$ ,  $\hat{\mathcal{F}}_2$  and  $\hat{\mathcal{F}}_3$  intersect each one another at a finite point. Let  $\hat{\mathcal{F}}_1 = \underline{ac}$ ,  $\hat{\mathcal{F}}_2 = \underline{ae}$ ,  $\hat{\mathcal{F}}_3 = \underline{ce}$  and  $\hat{\mathcal{F}}_4 = \underline{gh}$  be the four actuation forces. In turn, let  $\hat{\mathcal{M}}_{c1} = \underline{ij}$  and  $\hat{\mathcal{M}}_{c2} = \underline{kj}$  be the two constraint moments.

The corresponding superbracket is expressed and developed as follows:

$$S = [\underline{ac} \ \underline{ae} \ \underline{ce} \ \underline{gh} \ \underline{ij} \ \underline{kj}] = [\underline{aceh}] [\underline{acej}] [\underline{gikj}] \quad (7)$$

This sub-class of SMGs has three singularity conditions:

- (a)  $(\mathbf{m}_{c1} \times \mathbf{m}_{c2}) \cdot \mathbf{z} = 0$
- (b) plane  $(ace) \parallel \mathbf{f}_4$
- (c) plane  $(ace) \parallel \mathbf{z}$

### Superbracket Decomposition for Class B

#### B.1- Four skew lines where none of them is directed along $\mathbf{z}$

The extended Jacobian matrix of this sub-class of SMGs

takes the form:

$$\mathbf{J}_E^T = [\hat{\mathcal{F}}_1 \ \hat{\mathcal{F}}_2 \ \hat{\mathcal{F}}_3 \ \hat{\mathcal{F}}_4 \ \hat{\mathcal{M}}_{c1} \ \hat{\mathcal{M}}_{c2}] \quad (8)$$

where  $\hat{\mathcal{F}}_1$ ,  $\hat{\mathcal{F}}_2$ ,  $\hat{\mathcal{F}}_3$  and  $\hat{\mathcal{F}}_4$  correspond to four finite lines that are mutually skew and none of them is directed along  $\mathbf{z}$ . There exists (in a general configuration) a finite line  $\mathcal{L}_{ij}$  parallel to  $\mathbf{z}$  that crosses any couple of actuation forces  $\hat{\mathcal{F}}_i$  and  $\hat{\mathcal{F}}_j$ . Let  $\hat{\mathcal{F}}_1 = (\mathbf{f}_1, \mathbf{r}_1 \times \mathbf{f}_1)$ ,  $\hat{\mathcal{F}}_2 = (\mathbf{f}_2, \mathbf{r}_2 \times \mathbf{f}_2)$ ,  $\hat{\mathcal{F}}_3 = (\mathbf{f}_3, \mathbf{r}_3 \times \mathbf{f}_3)$  and  $\hat{\mathcal{F}}_4 = (\mathbf{f}_4, \mathbf{r}_4 \times \mathbf{f}_4)$  be the four actuation forces. In turn, let  $\hat{\mathcal{M}}_{c1} = \underline{\mathbf{i}}\underline{\mathbf{j}}$  and  $\hat{\mathcal{M}}_{c2} = \underline{\mathbf{k}}\underline{\mathbf{j}}$  be the two constraint moments. The superbracket then is depicted and developed as:

$$\begin{aligned} S &= [\underline{\mathbf{a}}\underline{\mathbf{b}} \ \underline{\mathbf{c}}\underline{\mathbf{d}} \ \underline{\mathbf{e}}\underline{\mathbf{f}} \ \underline{\mathbf{g}}\underline{\mathbf{h}} \ \underline{\mathbf{i}}\underline{\mathbf{j}} \ \underline{\mathbf{k}}\underline{\mathbf{j}}] \\ &= [\underline{\mathbf{g}}\underline{\mathbf{i}}\underline{\mathbf{k}}\underline{\mathbf{j}}] \left( [\underline{\mathbf{a}}\underline{\mathbf{b}}\underline{\mathbf{d}}\underline{\mathbf{f}}] [\underline{\mathbf{c}}\underline{\mathbf{g}}\underline{\mathbf{h}}\underline{\mathbf{j}}] \right) \\ &= [\underline{\mathbf{g}}\underline{\mathbf{i}}\underline{\mathbf{k}}\underline{\mathbf{j}}] \left( (\underline{\mathbf{a}}\underline{\mathbf{b}}\underline{\mathbf{d}}) \wedge (\underline{\mathbf{c}}\underline{\mathbf{g}}\underline{\mathbf{j}}) \wedge (\underline{\mathbf{f}}\underline{\mathbf{h}}) \right) \end{aligned} \quad (9)$$

This sub-class of SMGs has two main singularity conditions:

- (a)  $(\mathbf{m}_{c1} \times \mathbf{m}_{c2}) \cdot \mathbf{z} = 0$ .
- (b)  $((\mathbf{z} \times \mathbf{u}_{cg}) \times (\mathbf{f}_1 \times \mathbf{f}_2)) \cdot (\mathbf{f}_4 \times \mathbf{f}_3) = 0$

where  $\mathbf{u}_{cg}$  is the unit vector of a finite line non-parallel to  $\mathbf{z}$  and crossing lines  $\mathcal{L}_{12}$  and  $\mathcal{L}_{34}$ .

## B.2- Four skew lines where one of them is directed along $\mathbf{z}$

The extended Jacobian matrix of this sub-class of SMGs takes the form:

$$\mathbf{J}_E^T = [\hat{\mathcal{F}}_1 \ \hat{\mathcal{F}}_2 \ \hat{\mathcal{F}}_3 \ \hat{\mathcal{F}}_4 \ \hat{\mathcal{M}}_{c1} \ \hat{\mathcal{M}}_{c2}] \quad (10)$$

where  $\hat{\mathcal{F}}_1$ ,  $\hat{\mathcal{F}}_2$ ,  $\hat{\mathcal{F}}_3$  and  $\hat{\mathcal{F}}_4$  correspond to four finite lines that are mutually skew and  $\hat{\mathcal{F}}_4$  is parallel to  $\mathbf{z}$ . We know that, in a general configuration, there exists a finite line  $\mathcal{L}_{ij}$  parallel to  $\mathbf{z}$  that crosses any couple of actuation forces among  $\hat{\mathcal{F}}_1$ ,  $\hat{\mathcal{F}}_2$  and  $\hat{\mathcal{F}}_3$ . Thus, let  $\mathcal{L}_{12} \parallel \mathbf{z} = \underline{\mathbf{a}}\underline{\mathbf{c}}$  be the finite line parallel to  $\mathbf{z}$  that crosses, in a general configuration,  $\hat{\mathcal{F}}_1$  and  $\hat{\mathcal{F}}_2$  at the finite points  $\underline{\mathbf{a}}$  and  $\underline{\mathbf{c}}$ , respectively. Thus, points  $\underline{\mathbf{a}}$ ,  $\underline{\mathbf{c}}$  and  $\underline{\mathbf{j}}$  are aligned.

Let  $\hat{\mathcal{F}}_1 = \underline{\mathbf{a}}\underline{\mathbf{b}}$ ,  $\hat{\mathcal{F}}_2 = \underline{\mathbf{c}}\underline{\mathbf{d}}$ ,  $\hat{\mathcal{F}}_3 = \underline{\mathbf{e}}\underline{\mathbf{f}}$  and  $\hat{\mathcal{F}}_4 = \underline{\mathbf{g}}\underline{\mathbf{j}}$  be the four actuation forces. In turn, let  $\hat{\mathcal{M}}_{c1} = \underline{\mathbf{i}}\underline{\mathbf{j}}$  and  $\hat{\mathcal{M}}_{c2} = \underline{\mathbf{k}}\underline{\mathbf{j}}$  be the two constraint moments.

The corresponding superbracket is expressed and simplified

as follows:

$$\begin{aligned} S &= [\underline{\mathbf{a}}\underline{\mathbf{b}} \ \underline{\mathbf{c}}\underline{\mathbf{d}} \ \underline{\mathbf{e}}\underline{\mathbf{f}} \ \underline{\mathbf{g}}\underline{\mathbf{j}} \ \underline{\mathbf{i}}\underline{\mathbf{j}} \ \underline{\mathbf{k}}\underline{\mathbf{j}}] \\ &= [\underline{\mathbf{g}}\underline{\mathbf{i}}\underline{\mathbf{k}}\underline{\mathbf{j}}] \left( [\underline{\mathbf{a}}\underline{\mathbf{b}}\underline{\mathbf{c}}\underline{\mathbf{j}}] [\underline{\mathbf{d}}\underline{\mathbf{e}}\underline{\mathbf{f}}\underline{\mathbf{j}}] \right) \\ &= [\underline{\mathbf{g}}\underline{\mathbf{i}}\underline{\mathbf{k}}\underline{\mathbf{j}}] \left( (\underline{\mathbf{a}}\underline{\mathbf{b}}\underline{\mathbf{j}}) \wedge (\underline{\mathbf{e}}\underline{\mathbf{f}}\underline{\mathbf{j}}) \wedge (\underline{\mathbf{c}}\underline{\mathbf{d}}) \right) \end{aligned} \quad (11)$$

This sub-class of SMGs has two singularity conditions:

- (a)  $(\mathbf{m}_{c1} \times \mathbf{m}_{c2}) \cdot \mathbf{z} = 0$ .
- (b)  $((\mathbf{f}_1 \times \mathbf{z}) \times (\mathbf{f}_3 \times \mathbf{z})) \cdot \mathbf{f}_2 = 0$

## B.3- Two pairs of intersecting actuation forces

The extended Jacobian matrix of this sub-class of SMGs has the form:

$$\mathbf{J}_E^T = [\hat{\mathcal{F}}_1 \ \hat{\mathcal{F}}_2 \ \hat{\mathcal{F}}_3 \ \hat{\mathcal{F}}_4 \ \hat{\mathcal{M}}_{c1} \ \hat{\mathcal{M}}_{c2}] \quad (12)$$

where  $\hat{\mathcal{F}}_1 = (\mathbf{f}_1, \mathbf{r}_1 \times \mathbf{f}_1) = \underline{\mathbf{a}}\underline{\mathbf{b}}$ ,  $\hat{\mathcal{F}}_2 = (\mathbf{f}_2, \mathbf{r}_1 \times \mathbf{f}_2) = \underline{\mathbf{a}}\underline{\mathbf{d}}$ ,  $\hat{\mathcal{F}}_3 = (\mathbf{f}_3, \mathbf{r}_3 \times \mathbf{f}_3) = \underline{\mathbf{e}}\underline{\mathbf{f}}$ ,  $\hat{\mathcal{F}}_4 = (\mathbf{f}_4, \mathbf{r}_4 \times \mathbf{f}_4) = \underline{\mathbf{e}}\underline{\mathbf{h}}$ ,  $\hat{\mathcal{M}}_{c1} = \underline{\mathbf{i}}\underline{\mathbf{j}}$  and  $\hat{\mathcal{M}}_{c2} = \underline{\mathbf{k}}\underline{\mathbf{j}}$ .

The corresponding superbracket is expressed and simplified as follows:

$$\begin{aligned} S &= [\underline{\mathbf{a}}\underline{\mathbf{b}} \ \underline{\mathbf{a}}\underline{\mathbf{d}} \ \underline{\mathbf{e}}\underline{\mathbf{f}} \ \underline{\mathbf{e}}\underline{\mathbf{h}} \ \underline{\mathbf{i}}\underline{\mathbf{j}} \ \underline{\mathbf{k}}\underline{\mathbf{j}}] \\ &= [\underline{\mathbf{e}}\underline{\mathbf{i}}\underline{\mathbf{k}}\underline{\mathbf{j}}] \left( [\underline{\mathbf{a}}\underline{\mathbf{b}}\underline{\mathbf{d}}\underline{\mathbf{f}}] [\underline{\mathbf{a}}\underline{\mathbf{e}}\underline{\mathbf{h}}\underline{\mathbf{j}}] \right) \\ &= [\underline{\mathbf{e}}\underline{\mathbf{i}}\underline{\mathbf{k}}\underline{\mathbf{j}}] \left( (\underline{\mathbf{a}}\underline{\mathbf{b}}\underline{\mathbf{d}}) \wedge (\underline{\mathbf{a}}\underline{\mathbf{e}}\underline{\mathbf{j}}) \wedge (\underline{\mathbf{f}}\underline{\mathbf{h}}) \right) \end{aligned} \quad (13)$$

This sub-class of SMGs has two main singularity conditions:

- (a)  $(\mathbf{m}_{c1} \times \mathbf{m}_{c2}) \cdot \mathbf{z} = 0$ .
- (b)  $((\mathbf{z} \times \mathbf{u}_{ae}) \times (\mathbf{f}_1 \times \mathbf{f}_2)) \cdot (\mathbf{f}_4 \times \mathbf{f}_3) = 0$

where  $\mathbf{u}_{ae}$  is the unit vector of the finite line joining points  $\underline{\mathbf{a}}$  and  $\underline{\mathbf{e}}$ .

## APPLICATIONS

### The Quadruperon PM (Class A.2)

The Quadruperon PM [7] illustrated in Fig. 2, is composed of three identical limbs:  $\mathcal{L}_i = \underline{\mathbf{P}}^i \underline{\mathbf{R}}_1^i \underline{\mathbf{R}}_2^i \underline{\mathbf{R}}_3^i$ ,  $i = 1, 2, 3$ , plus one limb  $\mathcal{L}_4 = \underline{\mathbf{P}}^4 \underline{\mathbf{R}}_1^4 \underline{\mathbf{R}}_2^4 \underline{\mathbf{R}}_3^4$ .

The actuated joint  $\underline{\mathbf{P}}^i$  of the  $i$ th limb ( $i = 1, 2, 3$ ) is directed along  $\mathbf{m}_i$  which is parallel to the axes of the limb's three  $\underline{\mathbf{R}}^i$ -joints such that  $\mathbf{m}_1 \parallel \mathbf{m}_3 \parallel \mathbf{x}$  and  $\mathbf{m}_2 \parallel \mathbf{y}$ . In turn, the actuated joint  $\underline{\mathbf{P}}^4$  of the fourth limb is directed along  $\mathbf{z}$  which is parallel to the axes of the limb's three  $\underline{\mathbf{R}}^i$ -joints. It is noteworthy that vectors  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{z}$  have fixed directions.

**TABLE 2.** ACTUATION WRENCH OF LIMBS FOR 3T1R PMs WITH ONE ACTUATED JOINT.

<i>dof</i>	Types	Actuated joint	Actuated wrench
4	Permutation of $\bar{P}\bar{R}\bar{R}\bar{R}$ (the P-joint being along $\mathbf{z}$ )	$\underline{P}$	A force directed along $\mathbf{z}$
		$\underline{\dot{R}}$	A force orthogonal to $\mathbf{z}$ and lying in the plane containing the axes of two unactuated $\dot{R}$ -joints
	Permutation of $\bar{P}\bar{P}\bar{R}\bar{R}$	$\underline{P}$	A force non orthogonal to the direction of the actuated $\underline{P}$ -joint, orthogonal to the direction of the actuated $\underline{P}$ -joint and lying in the plane containing the axes of two unactuated $\dot{R}$ -joints
		$\underline{\dot{R}}$	A force non parallel to $\mathbf{z}$ , orthogonal to the directions of two $\underline{P}$ -joints and crossing the axis of the unactuated $\dot{R}$ -joint
	Permutation of $\bar{P}\bar{P}\bar{P}\bar{R}$	$\underline{P}$	A force orthogonal to the directions of two $\underline{P}$ -joints and crossing the axis of the unactuated $\dot{R}$ -joint
		$\underline{\dot{R}}$	A moment corresponding to a line at infinity that does not pass through point $\underline{j} = (\mathbf{z}, 0)$
5	Permutation of $\dot{R}\dot{R}\dot{R}\dot{R}$	$\underline{\dot{R}}$	A force collinear with the intersection line of two planes: the first one containing the axes of two $\dot{R}$ -joints and the second one containing the axes of two unactuated $\dot{R}$ -joints
		$\dot{R}$	A force parallel to $\mathbf{z}$ and intersecting the axis of unactuated $\dot{R}$ -joint
	Permutation of $\dot{R}\dot{R}\dot{R}\dot{R}$	$\underline{\dot{R}}$	A force parallel to $\mathbf{m}_i$ and intersecting the axis of the unactuated $\dot{R}$ -joint
		$\dot{R}$	A force collinear with the intersection line of two planes: one containing the axes of two $\dot{R}$ -joints and the second one containing the axes of two unactuated $\dot{R}$ -joints
	Permutation of $\bar{P}\bar{R}\dot{R}\dot{R}$ (the P-joint being along $\mathbf{z}$ )	$\underline{P}$	A force parallel to $\mathbf{z}$ and intersecting the axis of the $\dot{R}$ -joint
		$\underline{\dot{R}}$	The determination of the actuation wrench requires further details on the geometry of the limb
		$\dot{R}$	A force directed along $\mathbf{z}$
	Permutation of $\bar{P}\bar{R}\dot{R}\dot{R}$	$\underline{P}$	A force collinear with the intersection line of two planes: the first one containing the axes of two $\dot{R}$ -joints and the second one containing the axes of two $\dot{R}$ -joints
		$\underline{\dot{R}}$	The determination of the actuation wrench requires further details on the geometry of the limb
		$\dot{R}$	
	Permutation of $\bar{P}\bar{R}\dot{R}\dot{R}$ (the P-joint being along $\mathbf{m}_i$ )	$\underline{P}$	A force parallel to $\mathbf{m}_i$ and intersecting the axis of the $\dot{R}$ -joint
		$\underline{\dot{R}}$	A force parallel to $\mathbf{m}_i$
		$\dot{R}$	
	Permutation of $\bar{P}\bar{P}\dot{R}\dot{R}$	$\underline{P}$	
		$\underline{\dot{R}}$	
		$\dot{R}$	The determination of the actuation wrench requires further details on the geometry of the limb
	Permutation of $\bar{P}\bar{P}\dot{R}\dot{R}$	$\underline{P}$	
		$\underline{\dot{R}}$	
		$\dot{R}$	
	Permutation of $\bar{P}\bar{P}\bar{P}\dot{R}$	$\underline{P}$	
		$\underline{\dot{R}}$	A moment corresponding to a line at infinity that does not pass through point $\underline{j} = (\mathbf{z}, 0)$
		$\dot{R}$	A moment corresponding to a line at infinity that does not pass through point $\underline{m} = (\mathbf{m}_i, 0)$



**TABLE 3.** WRENCH GRAPHS AND SUPERBRACKET DECOMPOSITION FOR SMGs OF CLASSES A AND B.

sub-class	A.1	A.2
wrench graph		
superbracket decomposition	$[abdf][cikj][eikj]$	$[eikj]((abc) \wedge (bej) \wedge (hf))$
sub-class	A.3	B.1
wrench graph		
superbracket decomposition	$[aceh][acej][gikj]$	$[gikj]((abd) \wedge (cgj) \wedge (fh))$
sub-class	B.2	B.3
wrench graph		
superbracket decomposition	$[gikj]([abj] \wedge [efj] \wedge [cd])$	$[eikj]((abd) \wedge (aej) \wedge (fh))$

Limb  $\mathcal{L}_4$  is a 4-*dof* limb of Tab. 1. This limb is sufficient to make the Quadrupteron PM free of constraint singularities.

From Tab. 2, it follows that the four limbs of the Quadrupteron PM apply four actuation forces  $\hat{\mathcal{F}}_1 = (\mathbf{x}, \mathbf{r}_{A_1} \times \mathbf{x})$ ,

$\hat{\mathcal{F}}_2 = (\mathbf{y}, \mathbf{r}_{A_2} \times \mathbf{y})$ ,  $\hat{\mathcal{F}}_3 = (\mathbf{x}, \mathbf{r}_{A_3} \times \mathbf{x})$  and  $\hat{\mathcal{F}}_4 = (\mathbf{z}, \mathbf{r}_{A_4} \times \mathbf{z})$ . According to class A.2, the Quadrupteron PM exhibits an actuation singularity whenever:

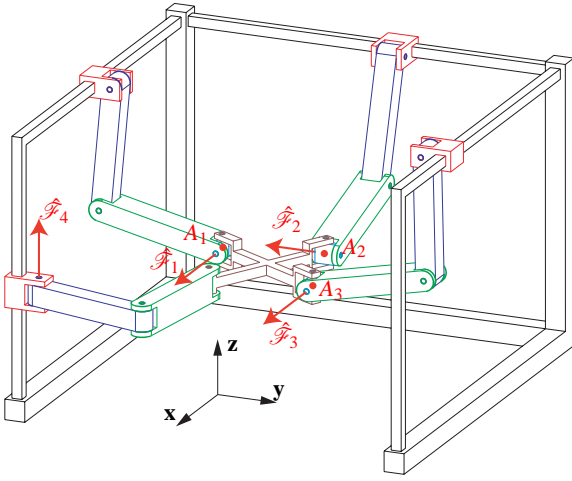


FIGURE 2. THE QUADRUPTERON PM.

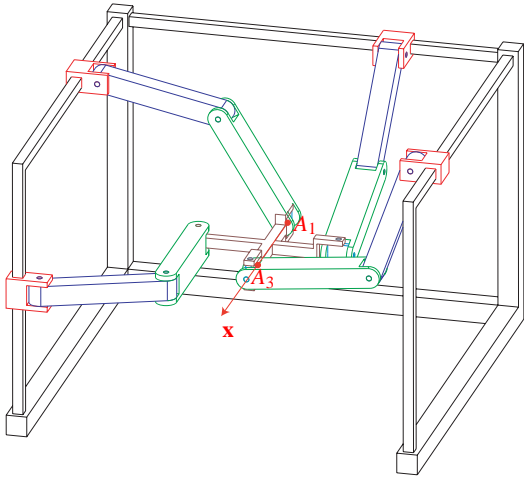


FIGURE 3. ACTUATION SINGULARITY OF THE QUADRUPTERON PM.

1.  $\mathbf{f}_1 \parallel \mathbf{u}_{ac}$
2.  $\mathbf{f}_1 \parallel \mathbf{z}$
3.  $\mathbf{f}_3 \parallel \mathbf{f}_4$

For the Quadrupter PM,  $\mathbf{f}_1 \equiv \mathbf{f}_3 \equiv \mathbf{x}$ ,  $\mathbf{f}_2 \equiv \mathbf{y}$ ,  $\mathbf{f}_4 \equiv \mathbf{z}$  and  $\mathbf{u}_{ac}$  is the unit vector of line  $A_1A_3$ . As a result, the second and the third singularity conditions are impossible. Consequently, the only possible singularity condition is:  $(A_1A_3 \parallel \mathbf{x})$ , which is illustrated in Fig. 3.

#### A 2-PRRU-PRRR-PRPP PM (class A.1)

Now let us modify the architecture of the Quadrupter PM in order to obtain a PM of the sub-class A.1. Accordingly, we should replace the third limb  $\mathcal{L}_3$  of the Quadrupter PM with a limb generating an actuation moment. From Tab. 2, the limb PRPP satisfies the required condition. As a result, the 2-PRRU-PRRR-PRPP PM, shown in Fig. 4, is obtained.

It can be noticed that the proposed PM is free of constraint singularities. The condition for actuation singularities is given

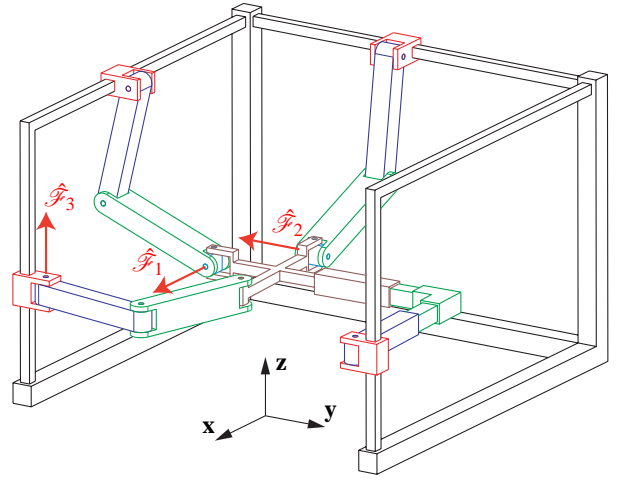


FIGURE 4. A 2-PRRU-PRRR-PRPP PM.

by:  $(\mathbf{f}_1 \times \mathbf{f}_2) \cdot \mathbf{f}_3 = 0$ . Since, for the proposed 2-PRRU-PRRR-PRPP PM,  $\mathbf{f}_1 \equiv \mathbf{x}$ ,  $\mathbf{f}_2 \equiv \mathbf{y}$  and  $\mathbf{f}_3 \equiv \mathbf{z}$ , this PM is also free of actuation singularities.

## CONCLUSION

The conceptual design of parallel Schnflies Motion Generators (SMGs) was addressed in this paper. The screw theory, the Grassmann-Cayley Algebra (GCA) and the Grassmann Geometry (GG) were used to define a procedure to consider the singularities at the conceptual design stage parallel SMGs.

First, we generated some conditions for the assembly of limbs. Then, the limbs likely to be used in the design of a parallel SMG were compared based on their influence on the constraint singularities of the obtained SMG. Moreover, the locus of the actuation wrench for each type of limbs and for each type of actuated joints was presented in Tab. 2. Based on screw theory and GG some rules for the validity of the selection of the actuated joints for parallel SMGs were generated. Accordingly, six types of wrench graph corresponding to six typical classes of parallel SMGs were developed. The geometric properties of these classes were highlighted in order to determine and compare the singularities of 3T1R PMs at the conceptual design stage. Finally, the results were applied to one PM of each class.

As a result, it is possible to generate new architectures of parallel SMGs by using:

- (i) the six typical wrench graphs with known singularity conditions;
- (ii) the comparison of limbs likely to be used in the design of a parallel SMG based on their influence on its constraint singularities;
- (iii) the locus of the actuation force for each choice of limb presented in Tab. 2.

The proposed approach developed for parallel SMGs can be applied to consider the singularities at the conceptual design stage of other classes of lower-mobility PMs.

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